A formal proof system for infinitary rational expressions

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by

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Abstract : We propound a new approach for studying infinitary rational languages and, by the way, we provide a formal system for proving the equality and emptiness of the intersection of infinitary rational languages.

Résumé : Nous présentons une nouvelle approche pour l'étude des langages rationnels infinitaires et nous en déduisons un système formel de preuve pour l'égalité et la vacuité de l'intersection des langages rationnels infinitaires.

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0. INTRODUCTION

Infinitary languages over a finite alphabet I_r provide an adequate mathematical model of the notion of machine behaviours, either finite or infinite. Among infinitary languages, the class of rational languages is of peculiar interest since rational languages are realized by finite transition systems (AN). The set of words recognized by a transition system is defined much the same way as for finite automata :

- a finite word f of $(I_r)^*$ is recognized by a transition system if and only if there exists a computation sequence, started from some initial configuration, which ends in a final configuration after than f has been red,
- an infinite word f of $(I_r)^\omega$ is recognized if and only if there exists an infinite computation sequence, started from some initial configuration, which reads f by passing infinitely often through a given set of distinguished configurations, called the infinitary configurations.

That notion of recognition was introduced by Büchi (Bu) and further studied by Mac-Naughton (Mn), Landweber (La) and Eilenberg (Ei). Nowadays, all this material developped in the 60's plays a crucial role in studies devoted to synchronization of processes, path expressions, proof of parallel programs and semantics of concurrency (see, among others, (AN) , (LTS), (DK)). It lays in the theoretical work about finite automata with infinite behaviours that classical decision problems such as the equality or emptiness have a positive answer for rational infinitary languages. It is thus of practical interest to develop tools for the computerized manipulation of infinitary automata.

However, a less practical approach is also worth value, and we shall present here, in the vein of Salomaa's theory of finitary rational expressions, a formal system for proving the equality (and emptiness of intersection) of infinitary rational expressions.

1. FINITARY AND INFINITARY RATIONAL EXPRESSIONS AND LANGUAGES

Let $I_r = \{x_1, \ldots, x_r\}$ be a finite alphabet with cardinality $r > o$. Elements x_j of I_r are called letters. A word over I_r is a finite or infinite sequence of letters. $W^\infty(I_r)$ denotes the set of words over I_r. For any f in $W^\infty(I_r)$, the length of f is written $\overline{f}$. $W(I_r)$ denotes the set of finite words over I_r, i.e. $W(I_r) = \{f \in W^\infty(I_r) \mid \overline{f} < \omega\}$. $W(I_r)$ is a partially ordered submonoid of $W^\infty(I_r)$ equipped with neutral element λ, catenation and partial order $\prec$ defined as follows :

- $\overline{\lambda} = 0$ (thus λ is the empty word) ;
- if $\overline{f} = \omega$ then $fg = f$, else $fg(n) = f(n)$ for $0 < n \leqslant \overline{f}$, $fg(\overline{f} + n) = g(n)$ for $0 < n \leqslant \overline{g}$, and $fg(i)$ is undefined elsewhere ;
- $f \prec g$ iff $\overline{f} \leqslant \overline{g}$ and $f(n) = g(n)$ for $0 < n \leqslant \overline{f}$.

A language over I_r is a finite or infinite subset of $W^\infty(I_r)$. The empty language is written L_ϕ. When applied to languages, the operator symbols $+$, $\cap$, $\sim$ and $\neg$ will respectively denote the set union, set intersection, complementation w.r.t. $W(I_r)$, and complementation w.r.t. $W^\infty(I_r)$. Other operators of interest are the catenation, iterative-closure (*) and omega-closure ($^\omega$) of languages, which are defined as follows :

- $UV = \{uv \mid u \in U, v \in V\}$
- $U^* = \sum_{i \in \mathbb{N}} U^i$, where $U^\circ = \{\lambda\}$ by convention
- if $U = \{\lambda\}$ then $U^\omega = \{\lambda\}$, else U^ω is the set of limits in $W^\infty(I_r) - W(I_r)$ of increasing sequences $(u_1 u_2 \ldots u_i)_{i \in \mathbb{N}}$ such that $(\forall i)\ (u_i \in U)$.

Now consider I_r as a graded alphabet whose symbols x_j have the arity 0. Let $I'' = \{+, {}^*, {}^\omega, \phi\}$ be an auxiliary alphabet whose symbols have respective arities 2,1,1,0 and let $I' = \{+, {}^*, \phi\}$. The set of infinitary rational expressions over I_r is the free word algebra $\mathbb{W}_{I_r + I''}$, and the set of finitary rational expressions over I_r is the free sub-algebra $\mathbb{W}_{I_r + I'}$. Each rational expression γ over I_r denotes a corresponding language $|\gamma|$ over I_r according to the following rules (Sa) :

- $|\phi| = L_\phi$
- $|x_j| = \{x_j\}$ for $0 < j \leqslant r$
- $|(\alpha + \beta)| = |\alpha| + |\beta|$
- $|(\alpha\ \beta)| = |\alpha|\ |\beta|$

$-|(\alpha^*)| = |\alpha|^*$, thus $|(\phi^*)| = \{\lambda\}$

$-|(\alpha^\omega)| = |\alpha|^\omega$, thus $|(\phi^\omega)| = L_\phi$

A language over I_r is infinitary rational iff it is denoted by an expression which belongs to $\mathbb{W}_{I_r + I''}$. A language over I_r is finitary rational iff it is denoted by an expression which belongs to $\mathbb{W}_{I_r + I'}$.

A classical theory of finitary rational expressions may be found in (Sa), where the author supplies a complete axiom system F for the equality in $\mathbb{W}_{I_r + I'}$ and shows that F can be extended into a complete axiom system for the equality in $\mathbb{W}_{I_r + I' + \{\sim,\cap\}}$ by simply adding derivation laws for the extra operators $\sim$ and $\cap$.

A comprehensive presentation of infinitary rational languages may be found in (Ei). The family of infinitary rational languages over I_r is closed for the intersection and complementation $\neg$, and coincides exactly with the family of languages which are denoted by expressions of the restricted form $\alpha + \sum_{i=1}^{n} \alpha_i \beta_i^\omega$, where α, α_i and β_i are members of $\mathbb{W}_{I_r + I'}$. Essential to our work is the Büchi - Mac Naughton theorem which assesses that for any language L denoted by an expression $\sum_{i=1}^{n} \alpha_i \beta_i^\omega$ with α_i and β_i as above, there exists a finite I_r - automaton $A_L = (Q,f,i,T)$, $T \subseteq Q$, and a finite deterministic I_r - automaton $A'_L = (Q',f',i',\mathcal{T}')$, $\mathcal{T}' \subseteq P(Q')$, such that $\| A_L \| = L = \| A'_L \|$ according to the following definitions :

$\| A_L \|$ is the set of the words which are accepted by A_L started from the initial state i along paths which pass infinitely many times through some particular state of T ;
$\| A'_L \|$ is the set of the words which are accepted by A'_L started from the initial state i' along paths such that the subset of states which are met infinitely many times is a member of $\mathcal{T}'$.

It lays between the lines of the proof of the above theorem that there exists a decision procedure for the equality in $\mathbb{W}_{I_r + I'' + \{\neg,\cap\}}$. What is lacking for infinitary rational expressions is the pendant of Salomaa's formal system for finitary rational expressions. Our purpose is precisely to construct such a system for infinitary expressions, building over the axiom system F of Salomaa with the help of the Büchi - Mac Naughton theorem. As was guessed beforehand, we have not succeeded in deriving $(\mathbb{W}_{I_r + I'' + \{\neg,\cap\}}, \{=\})$ from the restricted theory $(\mathbb{W}_{I_r + I''}, \{=\})$, the reason being that different deduction rules would be required for eliminating $\neg$ from $\neg\phi$ and $\neg(x_1^\omega)$ for instance. Nevertheless, let $\oplus$ denote the binary relation of disjointness of languages, then $(\mathbb{W}_{I_r} + I'' + \{\neg, \cap\}, \{=\})$ has exactly the same power as the theory $(\mathbb{W}_{I_r} + I'', \{=, \oplus\})$, for which we supply a complete proof system F^{∞} without induction rule. Our proof system is in fact a two-level proof system : the premisses of some of the deduction rules of F^{∞} include formulas such as $\alpha_s =_s \beta_s$, where subscript s means that the subscripted objects are to be considered in $(\mathbb{W}_{I_r} + I'', \{=\})$, so that $\alpha_s =_s \beta_s$ is valid in F^{∞} iff $\alpha = \beta$ is valid in $\boldsymbol{F}$.

The remaining of the paper is organized as follows. Section 2 recalls the axioms of Salomaa in order to make precise the exact relation $=_s$ which is used in the sequel. Section 3 brings in a first series of axioms and deduction rules of F^{∞} which allow to infer $\alpha = \beta$ or $\alpha \oplus \beta$ from premisses combining $=_s$ formulas and restricted formulas $\alpha' = \beta'$ or $\alpha' \oplus \beta'$ where α' and β' are expressions of the form $\sum_i \gamma_i \delta_i^{\omega}$. Section 4 presents the essence of our work : formulas of the above form are proved equivalent to similar formulas where $^{\omega}$ is replaced by $^{\Omega}$, meaning the infinite iteration over words of a language instead of the infinite iteration of the language itself ; taking advantage of that point, the equality or disjointness of infinitary rational languages may be shown to depend only on finite facts about corresponding finitary rational languages. The material developped there may be used either to devise decision procedures which do not follow the Büchi - Mac Naughton approach, or to construct a related proof system as it is undertaken in section 5. The final axiomatization is gathered in section 6.

2. THE AXIOMS OF SALOMAA

Let the following set of axiom schemes, where α, β and γ stand for any finitary rational expressions over I_r :

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \tag{1.1}$$

$$\alpha (\beta \gamma) = (\alpha \beta) \gamma \tag{1.2}$$

$$\alpha + \beta = \beta + \alpha \tag{1.3}$$

$$\alpha (\beta + \gamma) = \alpha \beta + \alpha \gamma \tag{1.4}$$

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma \tag{1.5}$$

$$\alpha + \alpha = \alpha \tag{1.6}$$

$$\phi^* \alpha = \alpha \tag{1.7}$$

$$\phi\alpha = \phi \tag{1.8}$$

$$\alpha + \phi = \alpha \tag{1.9}$$

$$\alpha^* = \phi^* + \alpha^* \alpha \qquad (1.10)$$

$$\alpha^* = (\phi^* + \alpha)^* \qquad (1.11)$$

$$\alpha \cap \beta = \sim ((\sim \alpha) + (\sim \beta)) \qquad (2.1)$$

$$\sim (\alpha_1 x_1 + \ldots + \alpha_r x_r) = (\sim \alpha_1) x_1 + \ldots + (\sim \alpha_r) x_r + \phi^* \qquad (2.2)$$

$$\sim (\alpha_1 x_1 + \ldots + \alpha_r x_r + \phi^*) = (\sim \alpha_1) x_1 + \ldots + (\sim \alpha_r) x_r \qquad (2.3)$$

It is proved in [Sa] that $(\mathbb{W}_{I_r} + I' + (\sim, \cap), \{=\})$ is the theory obtained from the above axioms together with the following inferences rules of replacement (R1) and deduction (R2) :

- R1 - Assume that X [ ] is a regular context so that X [α]is a rational expression, then from X [α] = γ and $\alpha = \beta$ one may infer X [β] = γ and X [α] = X [β]

- R2 - From $\alpha = \alpha \beta + \gamma$, one may infer $\alpha = \gamma \beta^*$ if β does not possess the empty word property (e.w.p.), that is β has none of forms δ^*, or $\delta_1 + \delta_2$ where either δ_1 or δ_2 possess e.w.p., or $\delta_1 \delta_2$ where both δ_1 and δ_2 possess e.w.p., or $\delta_1 \cap \delta_2$ where both δ_1 and δ_2 possess e.w.p., or $\sim \delta$ where δ does not possess e.w.p. .

From now on, $\alpha_s =_s \beta_s$, or more concisely $\alpha =_s \beta$, will be considered as a valid equation in $\overset{\infty}{F}$ if and only if $\alpha, \beta \in \mathbb{W}_{I_r} + I' + \{\sim, \cap\}$ and $\alpha = \beta$ is a valid equation in the system of Salomaa.

<u>Remark-</u> The theory of Salomaa might be presented more exactly as $(\mathbb{W}_{I_r} + I' + \{\sim, \cap\}$, $\{=$, e.w.p. , n.e.w.p.$\})$, where the additional properties e.w.p. and n.e.w.p. are characterized by extra axioms and inference rules. Additional properties and relations will be introduced similarly in our formal system for $(\mathbb{W}_{I_r} + I'' , \{=, \oplus\})$.

<u>3. THE DECOMPOSITION RULES</u>

The aim of the section is to supply $\overset{\infty}{F}$ with sufficient axioms and deduction rules for reducing the proofs of $\alpha = \beta$ or $\alpha \oplus \beta$ to corresponding proofs of restricted formulas $\alpha' \leq \beta'$ or $\alpha' \oplus \alpha''$, where α' and α'' (resp. β') are expressions of the form $\gamma_i \delta_i^\omega$ (resp. $\sum_i \gamma_i \delta_i^\omega$) such that γ_i and δ_i do not denote the empty language and belong to $\mathbb{W}_{I_r} + I' + \{\sim, \cap\}$, $|\delta_i|$ does not contain the empty word, and $|\gamma_i|$ does not contain the empty word or is equal to $\{\lambda\}$. The subsequent sections will concentrate on that restricted kind of statements.

The first step in the intended direction is to axiomatize the equivalence between general terms of $\mathbb{W}_{I_r} + I''$ and restricted expressions in Eilenberg's normal form $\beta + \sum_i \gamma_i \delta_i^{\omega}$ with $\beta \in W_{I_r} + I' + \{\sim, \cap\}$ and γ_i, δ_i as above. More precisely, for each term α of $W_{I_r} + I''$, the required axioms should allow to prove the equivalence between α and some corresponding expression α' in Eilenberg's normal form. Let properties ne (non empty) and new (non empty word) be defined upon infinitary rational languages over I_r as

- ne (α) iff $|\alpha| \neq L_{\phi}$
- new (α) iff $|\alpha| \neq \{\lambda\}$

Together with the axioms for ne and new, the following system A_1 fulfills the requirements. (α, β and γ stand for any term in $W_{I_r} + I''$).

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \qquad \text{A1.1}$$

$$\alpha \; (\beta \; \gamma) = (\alpha \; \beta) \; \gamma \qquad \text{A1.2}$$

$$\alpha + \beta = \beta + \alpha \qquad \text{A1.3}$$

$$\alpha \, (\beta + \gamma) = \alpha\beta + \alpha\gamma \qquad \text{A1.4}$$

$$(\alpha + \beta) \, \gamma = \alpha\gamma + \beta\gamma \qquad \text{A1.5}$$

$$\phi \, \alpha = \phi \qquad \text{A1.6}$$

$$\alpha \, \phi = \phi \qquad \text{A1.7}$$

$$\alpha + \phi = \alpha \qquad \text{A1.8}$$

$$(\phi^*)^{\omega} = \phi^* \qquad \text{A1.9}$$

$$\phi^{\omega} = \phi \qquad \text{A1.10}$$

$$\frac{\text{ne }(\beta),\ \text{new }(\beta),\ \beta + \phi^* = \gamma + \phi^*,\ \gamma =_s \gamma,\ \gamma \cap \phi^* =_s \phi}{\beta^{\omega} = \phi^* \gamma^{\omega}} \qquad \text{A1.11}$$

$$\frac{\text{new }(\alpha),\ \text{ne }(\beta)}{\alpha^{\omega}\beta = \alpha^{\omega}} \qquad \text{A1.12}$$

$$\frac{\text{new }(\delta)}{(\beta + \gamma\,\delta^{\omega})^* = \beta^* + \beta^*\,\gamma\,\delta^{\omega}} \qquad \text{A1.13}$$

$$\frac{\text{new }(\delta),\ \text{new }(\beta)}{(\beta + \gamma\,\delta^{\omega})^{\omega} = \beta^{\omega} + \beta^*\,\gamma\,\delta^{\omega}} \qquad \text{A1.14}$$

$$\frac{\text{ne }(\gamma),\ \text{ne }(\delta),\ \text{new }(\delta)}{(\phi^* + \gamma\,\delta^{\omega})^{\omega} = \gamma\,\delta^{\omega}} \qquad \text{A1.15}$$

$$\frac{\alpha =_s \beta}{\alpha = \beta} \qquad \text{R'1}$$

$$\frac{f(\alpha),\ \alpha = \beta}{f(\beta)} \quad \begin{cases} \text{for any formula context } f(\ ) \\ \text{constructed without using } =_s \end{cases} \qquad \text{R'2}$$

NB . By convention, R'1 and R'2 will be standard rules of every of the subsequently examined proof systems.

The consistency of the axioms and inference rules is obvious from the definition of the denotations of expressions. A simple induction on the structure of terms of $W_{I_r + I''}$, with sub-induction on the size of normal sum expressions, shows that any term may be proved equivalent in A_1 to a corresponding term in Eilenberg's normal form. Now, a completely adequate set of axioms for properties ne and new may be obtained from the following system A_2 (with R'1 and R'2 as implicit rules), so that our first step is completed :

$$\text{ne}(x_j) \quad 0 < j \leqslant r \qquad \text{A2.1} \qquad\qquad \text{new}(\phi) \qquad \text{A2.6}$$

$$\frac{\text{ne}(\alpha)}{\text{ne}(\alpha + \beta)} \qquad \text{A2.2} \qquad\qquad \text{new}(x_j) \quad 0 < j \leqslant r \qquad \text{A2.7}$$

$$\frac{\text{ne}(\alpha),\ \text{ne}(\beta)}{\text{ne}(\alpha\beta)} \qquad \text{A2.3} \qquad\qquad \frac{\text{new}(\alpha),\ \text{new}(\beta)}{\text{new}(\alpha + \beta)} \qquad \text{A2.8}$$

$$\text{ne}(\alpha^*) \qquad \text{A2.4} \qquad\qquad \frac{\text{new}(\alpha),\ \text{ne}(\alpha)}{\text{new}(\alpha + \beta)} \qquad \text{A2.9}$$

$$\frac{\text{ne}(\alpha)}{\text{ne}(\alpha^\omega)} \qquad \text{A2.5} \qquad\qquad \frac{\text{new}(\alpha)}{\text{new}(\alpha\beta)} \quad \frac{\text{new}(\alpha)}{\text{new}(\beta\alpha)} \qquad \text{A2.10,11}$$

$$\frac{\text{new}(\alpha),\ \text{ne}(\alpha)}{\text{new}(\alpha^*)} \qquad \text{A2.12} \qquad\qquad \frac{\text{new}(\alpha)}{\text{new}(\alpha^\omega)} \qquad \text{A2.13}$$

(The proofs of consistency and completeness are left to the reader).

In a second step, let us extend our current alphabet {=, $\oplus$, ne, new} of relation symbols over $W_{I_r + I''}$ by adding it the symbol of inclusion $\subseteq$ with conventional meaning. Consider the following set A_3 of axioms and deduction rules, where α, β, γ range over $W_{I_r} + I''$.

$$\frac{\alpha \subseteq \beta \ , \ \beta \subseteq \alpha}{\alpha = \beta} \qquad \text{A3.1}$$

$$\alpha \subseteq \alpha + \beta \qquad \text{A3.2}$$

$$\frac{\alpha \subseteq \gamma \ , \ \beta \subseteq \gamma}{\alpha + \beta \subseteq \gamma} \qquad \text{A3.3}$$

$$\frac{\alpha =_s \alpha \ , \ \beta =_s \beta, \ \alpha \cap (\sim\beta) =_s \phi}{\alpha \subseteq \beta} \qquad \text{A3.4}$$

$$\frac{\alpha \oplus \beta}{\beta \oplus \alpha} \qquad \text{A3.5}$$

$$\frac{\alpha \oplus \beta, \ \alpha \oplus \gamma}{\alpha \oplus (\beta + \gamma)} \qquad \text{A3.6}$$

$$\frac{\alpha =_s \alpha \ , \ \text{new } (\gamma)}{\alpha \oplus \beta\gamma^{\omega}} \qquad \text{A3.7}$$

$$\frac{\alpha =_s \alpha \ , \beta =_s \beta, \ \ \alpha \cap \beta =_s \phi}{\alpha \oplus \beta} \qquad \text{A3.8}$$

From the properties of $A_1 \cup A_2$, it is obvious that for any set B of axioms and inference rules, $A_1 \cup A_2 \cup A_3 \cup B$ is consistent and complete for $(W_{I_r} + I''$, {ne, new, = , $\subseteq$, $\oplus$}) if and only if B is consistent and complete for the restricted types of formulas $\alpha\ \beta^{\omega} \subseteq \sum_i \gamma_i\ \delta_i^{\omega}$ and $\alpha\ \beta^{\omega} \oplus\ \gamma_i\ \delta_i^{\omega}$ under the assumptions i) $\alpha =_s \alpha$, ne (α) ,$\alpha \cap \phi^* =_s \phi$ or $\alpha =_s \phi^*$ (and similarly for γ_i)
ii) $\beta =_s \beta$, ne (β) ,$\beta \cap \phi^* =_s \phi$ (and similarly for δ_i).

The remaining of the paper intends to find out an adequate system B which satisfies the above specification.

NB . Notice that $\alpha =_s \alpha$ is redundant in the above condition i).

4. ITERATION OVER WORDS OF A LANGUAGE VS. ITERATION OF THE LANGUAGE.

In this section, we shall leave for a moment our axiomatic apparatus and put forward some interesting facts about the infinite iteration of rational languages. It will precisely be shown that there exists very strong connections between the infinite iteration L^{ω} of a rational language L and the periodic iteration $(L^{+})^{\Omega}$ of L^{+}. The key propositions show that statements $\alpha\beta^{\omega} \subseteq \sum_i \gamma_i\ \delta_i^{\omega}$ and $\alpha\beta^{\omega} \oplus \gamma\delta^{\omega}$ are equivalent to similar statements where the operator $^{\omega}$ has been replaced by $^{\Omega}$. Although the use of the periodic iteration operator $^{\Omega}$ drives us outside the class of infinitary rational languages, the subsequent propositions tell that the trip may be ended by a return into the central class of finitary rational languages : both the iteration operator $^{\omega}$ and its fellow operator $^{\Omega}$ can be eradicated from the formulation of properties $\alpha \subseteq \beta$ and $\alpha \oplus \beta$.

Let us now work out the fundamental definition and propositions. In the text below, capital letters B and C will always stand for finitary rational languages over I_r. We take the implicit conventions that $\lambda \notin C \neq L_{\phi}$ and $B = \{\lambda\}$ or $\lambda \notin B \neq L_{\phi}$. These conventions hold everywhere in the section.

<u>Definition 1</u>. C^{Ω}, the periodic iteration of C, is the set of limits in $W^{\infty}(I_r)$ of increasing sequences $(u^i)_{i\in\mathbb{N}}$, for u ranging over C. By convention, $(L_{\phi})^{\Omega}$ equals L_{ϕ}.

<u>proposition 2.</u> $(C_1 + C_2)^{\Omega} = C_1^{\Omega} + C_2^{\Omega}$

(obvious)

<u>proposition 3.</u> $BC^{\omega} \subseteq \sum_i B_i\ C_i^{\omega}$ iff $BC^*(C^+)^{\Omega} \subseteq \sum_i B_i\ C_i^*(C_i^+)^{\Omega}$

- where C^+ is the usual notation standing for CC^* -

<u>proof of the only-if part</u>. Let $\phi \in BC^*$ and $\psi \in C^+$, then $\phi\psi^{\omega} \in BC^{\omega}$.By the hypothesis $BC^{\omega} \subseteq \sum_i B_i\ C_i^{\omega}$, there must occur some value j of the index i such that $\phi\psi^{\omega} \in B_j\ C_j^{\omega}$. From the assumptions $\lambda \notin C_j \neq L_{\phi}$ and $B_j = \{\lambda\}$ or $\lambda \notin B_j \neq L_{\phi}$, and according to the Büchi- Mac Naughton theorem, there exists a finite I_r-automaton a with a single infinitary state q^+, let $a = (Q, f, q_0, \{q^+\})$, such that $B_jC_j^{\omega} = \|a\|$, $B_jC_j^* = |a|$, and $C_j^* = |(Q, f, q^+, \{q^+\})|$.

Let $q \xrightarrow{x_i} q'$ iff $q' \in f(q,x_i)$, and let $\Longrightarrow$ denote the reflexive and transitive closure of $\rightarrow$ so that $q \Longrightarrow q$ and, for $m > 0$, $q \xRightarrow{x_{k1}\ x_{k2}\cdots\ x_{km}} q'$ holds iff $m = 1$ and $q \xrightarrow{x_{k1}} q'$ or $q \xrightarrow{x_{k1}} q''$ and $q'' \xRightarrow{x_{k2}\ \cdots\ x_{km}} q'$ for some q'' in Q.

Put $\psi = \psi_1 \psi_2 \cdots \psi_n$, $\psi_k \in I_r$, then $\lambda \notin C_j$ implies that $n > 0$. Since $\phi \psi^\omega$ belongs to $\| a \|$, there certainly exists an infinite path

$$\begin{tikzcd}
q_0 \arrow[r, Rightarrow, "\phi"] & q_1 \arrow[rrrr, Rightarrow, "\psi"] \arrow[dr, "\psi 1"'] & & & & q_2 \arrow[rrrr, Rightarrow, "\psi"] \arrow[dr, "\psi 1"'] & & & & q_3 \cdots \\
& & q_{12} \arrow[r, "\psi 2"'] & q_{13} \cdots \arrow[r, "\psi n-1"'] & q_{1n} \arrow[ur, "\psi n"'] & & q_{21} \arrow[r, "\psi 2"'] & q_{23} \cdots \arrow[r, "\psi n-1"'] & q_{2n} \arrow[ur, "\psi n"'] &
\end{tikzcd}$$

which passes infinitely many times through the infinitary state q^+. For $i > 0$, put $q_{i1} = q_i$, and let R be the binary relation over $\mathbb{N}_+$ defined by $R(i,j)$ iff $q^+ \in \{q_{kl} \mid i \leqslant k < j,\ 1 \leqslant l \leqslant n\}$. The relation R is obviously transitive, and for any positive i, the R-image of i is an infinite set. Since Q is a finite set of states, it clearly follows that there exists some pair (i,j) of integers in R for which $q_i = q_j$ holds. Let q_{kl} be the first occurence of the infinitary state q^+ in the corresponding sequence of states $q_{i1}\ q_{i2} \cdots q_{in}\ q_{i+1,1} \cdots q_{j-1,n}\ q_j$. Put $\phi' = \phi \psi^{k-1} \psi_1 \psi_2 \cdots \psi_{l-1}$ and $\psi' = \psi_l \cdots \psi_n \psi^{j-i-1} \psi_1 \cdots \psi_{l-1}$, then $\phi' \in B_j C_j^*$, $\psi' \in C_j^+$ and the equality $\phi \psi^\omega = \phi' \psi'^\omega$ is verified. Thus, $\phi\psi^\omega \in \sum_i B_i C_i^* (C_i^+)^\Omega$ as it was to demonstrate.

<u>proof of the if part</u>. let $\phi = \phi_0\ \phi_1\ \phi_2 \cdots \phi_j \ldots \in BC^\omega$ with ϕ_0 in B and ϕ_j in C for $j > 0$, thus $\phi_j \neq \lambda$. One has to prove that ϕ belongs to $\sum_i B_i C_i^\omega$ under the hypothesis $BC^*(C^+)^\Omega \subseteq \sum_i B_i C_i^* (C_i^+)^\Omega$. From the assumptions $\lambda \notin C_i \neq L_\phi$, and according to the Büchi- Mac Naughton theorem, there exists a finite deterministic I_r-automaton a with a set of subsets of infinitary states, let $a = (Q, f, q_0, \{E_1, E_2, \ldots, E_k\})$, such that $\sum_i B_i C_i^\omega = \| a \|$. For any strictly positive integer k, $\phi_0\ \phi_1\ \phi_2 \ldots \phi_k (\phi_k)^\omega$ is a word of $BC^*(C^+)^\Omega$ and thus also a word of $\sum_i B_i C_i^\omega$ from the obvious inclusion $\sum_i B_i C_i^* (C_i^+)^\Omega \subseteq \sum_i B_i C_i^\omega$. Since the above property holds for any $k > 0$, it follows from the determinism of a that there certainly exists an infinite path $q_0 \xRightarrow{\phi 0} q_1 \xRightarrow{\phi 1} q_2 \cdots \xRightarrow{\phi k-1} q_k \cdots$ such that $q_{i-1} \xRightarrow{\phi i} q_i$ holds in a for any strictly positive i. Let q be one of the states which occur infinitely many times in the sequence $q_0\ q_1\ q_2 \cdots q_k \ldots$, and let $(i_j)_{j>0}$ be the increasing sequence of the integers which verify $q_{i_j} = q$. Put $\phi'_0 = \phi_0\ \phi_1 \ldots \phi_{i_1 - 1}$, and for $j > 0$, put $\phi'_j = \phi_{i_j}\ \phi_{i_j + 1} \ldots \phi_{i_{j+1} - 1}$. Obviously, $\phi'_0 \in BC^*$ and $\phi'_j \in C^+$ for $j > 0$.

Consider the a-transition sequence $S = q_0 \xRightarrow{\phi' 0} q \xRightarrow{\phi' 1} q \ldots \xRightarrow{\phi' k} q \ldots$
For $j > 0$, let E_j be the subset of the states, including q, which are met in the course of the transitions $q \xRightarrow{\phi' j} q$. By the hypothesis $BC^*(C^+)^\Omega \subseteq \sum_i B_i C_i^* (C_i^+)^\Omega$, $\phi'_0\ \phi'_1 \cdots \phi'_{j-1} (\phi'_j)^\omega$ is a word of $\sum_i B_i C_i^* (C_i^+)^\Omega$ and thus also a word of

$\sum_i B_iC_i^{\omega}$ from the obvious inclusion $\sum_i B_iC_i^*(C_i^+)^{\Omega} \subseteq \sum_i B_iC_i^{\omega}$; since $\sum_i B_iC_i^{\omega} = \| \alpha \|$, it follows from the determinism of α that E_j is one of the elements of the set $\{E_1, \ldots, E_k\}$, let $E_j = E_{n_j}$. Define $\{l_1, \ldots, l_m\}$ as the set of the integers which occur infinitely many times in $(n_j)_{j>0}$. It is easily seen that the subset of the states which are met infinitely many times in the course of the sequence S is exactly the union of subsets $E_{l_1} \cup E_{l_2} \ldots \cup E_{l_m}$.

Of course, one can always find a pair of integers (q,r) which verifies $\{l_1,\ldots,l_m\} = \{n_j \mid q \leqslant j < r\}$. Now, the infinite word $\phi'_0\, \phi'_1 \ldots \phi'_{q-1} (\phi'_q \ldots \phi'_{r-1})^{\omega}$ obviously belongs to $BC^*(C^+)^{\Omega}$, and therefore to $\sum_i B_iC_i^{\omega}$ for the same reasons as above : the determinism of α makes it clear that the union of subsets $E_{l_1} \cup E_{l_2} \ldots \cup E_{l_m}$

is itself an element of the set $\{E_1, \ldots, E_k\}$.

The above properties allow to draw the conclusion

$\phi'_0\, \phi'_1 \ldots \phi'_k \ldots \in \sum_i B_iC_i^{\omega}$, that is still $\phi \in \sum_i B_iC_i^{\omega}$ which was to be demonstrated. ■

<u>proposition 4.</u> $BC^{\omega} \cap B'C'^{\omega} = L_{\phi}$ iff $BC^*(C^+)^{\Omega} \cap B'C'^*(C'^+)^{\Omega} = L_{\phi}$

<u>proof</u> Let L= $BC^{\omega} \cap B'C'^{\omega}$. Since L is also an infinitary rational language, it follows from the assumptions $\lambda \notin C$ and $\lambda \notin C'$ that L may always be written as a (possible empty) sum $\sum_i B''_i C''^{\omega}_i$. An equivalent formulation of $BC^{\omega} \cap B'C'^{\omega} \neq L_{\phi}$ is therefore $(\exists B'')\ (\exists C'')\ (B''C''^{\omega} \subseteq BC^{\omega}$ and $B''C''^{\omega} \subseteq B'C'^{\omega})$,
which still amounts by proposition 3 to
$(\exists B'')\ (\exists C'')\ (B''C''^*(C''^+)^{\Omega} \subseteq BC^*(C^+)^{\Omega}$ and $B''C''^*(C''^+)^{\Omega} \subseteq B'C'^*(C'^+)^{\Omega})$

- recall our general convention that B's and C's differ from L_{ϕ} and that C's do not contain λ -

Suppose that $BC^*(C^+)^{\Omega} \cap B'C'^*(C'^+)^{\Omega} = L_{\phi}$, then the disjointness of BC^{ω} and $B'C'^{\omega}$ is clearly implied.

Suppose that $BC^*(C^+)^{\Omega} \cap B'C'^*(C'^+)^{\Omega} \neq L_{\phi}$, then the non disjointness of BC^{ω} and $B'C'^{\omega}$ is made obvious by the inclusions $BC^*(C^+)^{\Omega} \subseteq BC^{\omega}$ and $B'C'^*(C'^+)^{\Omega} \subseteq B'C'^{\omega}$. ■

The end of the section aims at providing equivalent finitary formulations of the statements $BC^*(C^+)^{\Omega} \subseteq \sum_i B'_i C'^*_i (C'^+_i)^{\Omega}$ and $BC^*(C^+)^{\Omega} \cap B'C'^*(C'^+)^{\Omega} = L_{\phi}$. The task will be cut into successive steps :

i) eliminate the left factor BC^* from the above statements, then
ii) eliminate the remaining primed factors $B'C'^*$, and finally
iii) prove that the resulting relations $(C^+)^\Omega \subseteq (D)^\Omega$ and $(C^+)^\Omega \cap (D)^\Omega = L_\phi$ can be precisely characterized by finitary facts about C and D. The proofs of the propositions are given in the appendix. Henceforth, Rat (I_r^*) will denote the class of finitary languages over I_r. Before the propositions can be expressed, some preliminary definitions of auxiliary operators upon $\mathrm{Rat}(I_r^*)$ are needed.

definition 5. For any language L in $\mathrm{Rat}(I_r^*)$ and for any word ϕ in I_r^*, the ϕ-residue of L is the language L/ϕ equal to $\{\psi \in I_r^* \mid \phi\psi \in L\}$

A well known property of $\mathrm{Rat}(I_r^*)$ indicates that the set of residues of a rational language is a finite set of rational languages.

definition 6. For any n-dimensional vector $\langle L_1, \ldots, L_n\rangle$ of languages L_i in Rat (I_r^*), for any word ϕ in I_r^*, and for any language L in $\mathrm{Rat}(I_r^*)$:

$\langle L_1, \ldots, L_n\rangle / \phi$ is the n-dimensional vector $\langle L_1/\phi, \ldots, L_n/\phi\rangle$,
$\langle L_1, \ldots, L_n\rangle \div L$ is the set of vectors $\{\langle L_1, \ldots, L_n\rangle/\phi \mid \phi \in L\}$.

Given L_i as above and $L \neq L_\phi$, $\langle L_1, \ldots, L_n\rangle \div L$ is obviously a non empty and finite set of vectors with elements in Rat (I_r^*).

definition 7. For any pair of languages L and L' in Rat (I_r^*), $L \,\%\, L'$ is the language given by $\{\psi \in I_r^* \mid (\exists\, \phi \in L')\ (\phi\psi \in L)\}$.

definition 8. For any pair of languages L and L' in Rat (I_r^*), $L \sim L'$ is the language given by $\{\phi\psi \mid \phi \in L' \text{ and } \psi\phi \in L\}$.

The rationality of operators $\%$ and $\sim$ will be proved in section 5, and may consequently be assumed.

proposition 9. Let $\langle B_1C_1^*, \ldots, B_nC_n^*\rangle \div BC^*$ be equal to $\{\langle X_{11}, \ldots, X_{n1}\rangle, \ldots, \langle X_{1m}, \ldots, X_{nm}\rangle\}$, so that $X_{ij} = X_{ij}\,C_i^*$, then $BC^*(C^+)^\Omega \subseteq \sum_i B_iC_i^*\,(C_i^+)^\Omega$ holds if and only if $(C^+)^\Omega \subseteq \sum_i X_{ij}\,C_i^*\,(C_i^+)^\Omega$ holds for any j s.t. $1 \leqslant j \leqslant m$.

proposition 10. $BC^*(C^+)^\Omega \cap B'C'^*(C'^+)^\Omega = L_\phi$ if and only if $(C^+)^\Omega \cap ((B'C'^*) \,\%\, (BC^*))\,(C'^+)^\Omega = L_\phi$, or yet equivalently $(C^+)^\Omega \cap ((B'C'^*) \,\%\, (BC^*))\, C'^*(C'^+)^\Omega = L_\phi$.

<u>proposition 11.</u> Given languages A_i in $\mathrm{Rat}(I_r^*)$, $1 \leqslant i \leqslant n$, $(C^+)^\Omega \subseteq \sum_i A_i C_i^* (C_i^+)^\Omega$ holds if and only if
$(C^+)^\Omega \subseteq (\sum_i (C_i^+ \,\backsim\, A_i C_i^*))^\Omega$

- notice that the second member can be written as D^Ω where $\lambda \notin D$.

<u>proposition 12.</u> Given A' in Rat (I_r^*), $(C^+)^\Omega \cap A'C'^*(C'^+)^\Omega = L_\phi$ if and only if
$(C^+)^\Omega \cap (C'^+ \,\backsim\, A'C'^*)^\Omega = L_\phi$

<u>definition 13.</u> Let L in Rat (I_r^*), L possesses the cycle-free property cf (L) if and only if $(\forall \phi \in L)\ (\exists\, p > 0)\ (\phi^p \notin L)$.

<u>proposition 14.</u> Given languages A_i in $\mathrm{Rat}(I_r^*)$, $1 \leqslant i \leqslant n$, $(C^+)^\Omega \subseteq (\sum_i (C_i^+ \,\backsim\, A_i C_i^*))^\Omega$ holds if and only if $\mathrm{cf}(C^+ \cap (\sim(\sum_i (C_i^+ \,\backsim\, A_i C_i^*))))$

- where $\sim$ denotes the complementation w.r.t. I_r^* -

<u>proposition 15.</u> Given A' in $\mathrm{Rat}(I_r^*)$, $(C^+)^\Omega \cap (C'^+ \,\backsim\, A'C'^*)^\Omega = L_\phi$ if and only if $C^+ \cap (C'^+ \,\backsim\, A'C'^*) = L_\phi$

At the present time, we have proved that the critical properties $BC^\omega \subseteq \sum_i B_i C_i^\omega$ and $BC^\omega \cap B'C'^\omega = L_\phi$ can be equivalently restated in terms of the cycle-free and emptiness properties of some related finitary rational languages. This intermediate result may be incorporated into the design of a decision procedure for infinitary rational languages, provided that effective procedures are found for implementing the operations $\div$, $\%$, $\backsim$ and the test of the cycle-free property, at the level of standard finite-state automata. Since our explicit intention is to obtain a formal proof system for infinitary rational languages, we shall slightly depart from the approach we have just sketched : rather than searching for effective procedures, next section does prepare the way for axiomatizing the operators $\div$, $\%$, $\backsim$ and the cf property.

5. SOME AUXILIARY PROPERTIES OF FINITARY RATIONAL LANGUAGES.

Only finitary rational languages over I_r will be considered here, and we therefore adopt the convention that variables represented by capital letters range implicitly over $Rat(I_r^*)$. The first half of the section states some rather general properties of rational languages. Those general properties are used in the second half to obtain alternative characterizations of the cf-property and operators $\div$, $\dot{/}$, $\frown$ as defined in sect. 4. The proofs of the propositions are given in the appendix.

Some general properties of rational languages .

definition 16. Given infinitary rational languages L and L' over I_r , L is the set of proper prefixes of L' $(L \; \pi \; L')$ if and only if $L' \in Rat(I_r^*)$ and $L = \{\phi \in I_r^* \mid (\exists \psi \in I_r^*) \; (\psi \neq \lambda \text{ and } \phi \psi \in L')\}$.

proposition 17. The following implications universally hold :

- $L = L_\phi \supset L_\phi \; \pi \; L$
- $L = \{x_j\} \supset \{\lambda\} \; \pi \; L$
- $L_1 \; \pi \; L'_1$ and $L_2 \; \pi \; L'_2 \supset (L_1 + L_2) \; \pi \; (L'_1 + L'_2)$
- $L_1 \; \pi \; L'_1$ and $L_2 \; \pi \; L'_2$ and $L'_2 \neq L_\phi \supset (L_1 + L'_1 L_2) \; \pi \; (L'_1 L'_2)$
- $L \; \pi \; L' \supset (L'^* L) \; \pi \; (L'^*)$

(the proof is immediate, and is left to the reader).

definition 18. Given L' in $Rat(I_r^*)$, L' has the prefix-free property $pf(L')$ if and only if $L \; \pi \; L'$ and $L \cap L' = L_\phi$ for some L.

proposition 19. Given L in $Rat(I_r^*)$, there exist finite families of prefix-free languages K_i, K'_i such that $L = \sum_i K_i \, K'^*_i$ - prefix normal form-
(the proof of this classical property may be found in [Ei]).

definition 20. Given L in $Rat(I_r^*)$, X is a co-residue of L if and only if X equals $\{\phi \in I_r^* \mid (L/\phi) = X'\}$ for some non empty residue X' of L. X and X' are said L-correlated when they satisfy the above condition.

proposition 21. Given L in $Rat(I_r^*)$, let language X be a co-residue of L, then X belongs to $Rat(I_r^*)$.

<u>proposition 22.</u> Let L' in $\mathrm{Rat}(I_r^*)$ and let finite families of languages X_i, X'_i in $\mathrm{Rat}(I_r^*)$, $1 \leqslant i \leqslant n$. If $\{X'_1, \ldots, X'_n\}$ is the set of non-empty residues of L' and if X_i, X'_i are L'-correlated for any i, then the following equality holds :

$$L' = \sum_{i=1}^{n} X_i X'_i \quad \text{- residual normal form-}$$

<u>proposition 23.</u> Given L' in $\mathrm{Rat}(I_r^*)$, let X_i, X'_i in $\mathrm{Rat}(I_r^*)$ such that $L' = \sum_{i=1}^{n} X_i X'_i$, then the conditions of the above proposition are fulfilled if and only if the following assertions are valid :

- $(\forall i)\ (X_i \neq L_\phi \text{ and } X_i\, (\sim X'_i) \cap L' = L_\phi)$
- $(\forall i)\ (\forall j)\ (i \neq j \supset X'_i \neq X'_j)$
- $(\exists L)\ (L\ \pi\ L' \text{ and } L + L' = \sum_{i=1}^{n} X_i)$

(the proofs of the last two propositions are left to the reader).

Some features of the operators ÷ , ·/. , ∞ and cf property.

<u>proposition 24.</u> Given B and B_i in $\mathrm{Rat}(I_r^*) - L_\phi$, $i=1 \ldots n$, $\langle B_1, \ldots, B_n \rangle \div B = \{\langle X_{11}, \ldots, X_{n1} \rangle, \ldots, \langle X_{1m}, \ldots, X_{nm} \rangle\}$ if and only if the following cond. 1 to 3 are verified for some family of rational languages Z_j, $j=1 \ldots m$:

1. $(\forall j)\ (Z_j \neq L_\phi)$
2. $\sum_j Z_j = B$
3. $(\forall i)\ (\forall j)\ (Z_j X_{ij} \subseteq B_i \text{ and } Z_j\, (\sim X_{ij}) \cap B_i = L_\phi)$

<u>proposition 25.</u> Given B and B' in $\mathrm{Rat}\,(I_r^*)$, $B' \neq L_\phi$, B' ·/. $B = Z$ if and only if there exist finite families of rational languages X_i, Y_i and Z_i, $i=1 \ldots n$, such that the following conditions are verified :

- $(\forall i)\ (X_i \neq L_\phi \text{ and } X_i Y_i \subseteq B' \text{ and } X_i\, (\sim Y_i) \cap B' = L_\phi)$
- $(\exists L)\ (L\ \pi B' \text{ and } L + B' = \sum_i X_i)$
- $(\forall i)\ ((X_i \cap B = L_\phi \text{ and } Z_i = L_\phi) \text{ or } (X_i \cap B \neq L_\phi \text{ and } Z_i = Y_i))$
- $Z = \sum_i Z_i$

(immediate from the definition of ·/. and proposition 23) proposition 26. Given B and B' in Rat (I_r^*), $B' \neq L_\phi$, $B' \approx B = Z$ if and only if there exist finite families of rational languages X_i and Y_i, $i=1 \ldots n$, such that the following conditions are verified :

- $(\forall i)$ $(X_i \neq L_\phi$ and $X_i\ Y_i \subseteq B'$ and $X_i\ (\sim Y_i) \cap B' = L_\phi)$

- $(\exists L)$ $(L\ \pi B'$ and $L+B' = \sum_i X_i)$

- $Z = \sum_i (Y_i \cap B)\ X_i$

(immediate from the definition of $\approx$ and proposition 23).

proposition 27. Given X and Y in Rat (I_r^*), cf $(X + Y)$ holds if and only if cf (X) and cf (Y) both hold.

proposition 28. Given prefix-free languages X and Y in Rat (I_r^*), cf (XY^*) holds if and only if $XY^* \cap Y^* = L_\phi$.

6. THE PROOF SYSTEM

It is now a simple exercise in formulation to tie together the various results of sections 4 and 5 into a formal system B, which fulfills the requirements expressed in section 3. The only inventive part is to add auxiliary relation symbols for keeping the inference rules within a reasonable size. The following relations over infinitary languages will be used without underlining their associated symbols :

- $\underline{pf}\ (B) \overset{def}{\Longleftrightarrow} B \in Rat(I_r^*)$ and pf (B)
- $\underline{cf}\ (B) \overset{def}{\Longleftrightarrow} B \in Rat(I_r^*)$ and cf (B)
- $(B,C)\ \underline{\approx}\ D \overset{def}{\Longleftrightarrow} B,C,D \in Rat(I_r^*)$ and $B \neq L_\phi$ and $D = B \approx C$
- $(B,C)\ \underline{\%}\ D \overset{def}{\Longleftrightarrow} B,C,D \in Rat(I_r^*)$ and $B \neq L_\phi$ and $D = B \,\%\, C$

B is made of six families of rules, together with the implicit rules R'1 and R'2 as defined in the above section 3. Unhappily, some of the six families are countably infinite sets of inference rules : we are left with no other choice than resorting to corresponding schemes of inference rules. An explanatory example, not included in B, is for instance the scheme

$$\frac{(\forall i \in (n)\ (\alpha_i = \beta_i \text{ or } \alpha_i = \phi)}{\sum_{i=1}^{n} \alpha_i \subseteq \sum_{i=1}^{n} \beta_i}$$

which generates the infinite sequence of inference rules

$$\frac{\alpha_1 = \beta_1}{\alpha_1 \subseteq \beta_1}\ ,\ \frac{\alpha_1 = \phi}{\alpha_1 \subseteq \beta_1}\ ,\ \frac{\alpha_1 = \beta_1\ , \alpha_2 = \beta_2}{\alpha_1 + \alpha_2 \subseteq \beta_1 + \beta_2}\ ,\ \frac{\alpha_1 = \beta_1\ , \alpha_2 = \phi}{\alpha_1 + \alpha_2 \subseteq \beta_1 + \beta_2}\ \ldots$$

with this convention in mind, let us now turn to the detailed presentation of the rules.

The inclusion rules (B_1)

Let S_0 to S_6 respectively stand for the following schemes of (finite) sets of formulas :

$$S_0 \equiv (\forall j \in (o,n))$$
$$(\text{ ne } (\alpha_j) \ \underline{\text{and}}\ ((\alpha_j \cap \phi^* =_s \phi)\ \underline{\text{or}}\ (\alpha_j =_s \phi^*))\underline{\text{and}} \text{ ne } (\beta_j)\ \underline{\text{and}}\ (\beta_j \cap \phi^*) =_s \phi\ )$$

$$S_1 \equiv (\alpha_0\ \beta_0^{\ *} =_s \sum_{j=1}^{m} \delta_j)$$

$$S_2 \equiv (\forall j \in (m))\ (\text{ne } (\delta_j))$$

$$S_3 \equiv (\forall i \in (n))\ (\forall j \in (m))$$
$$((\delta_j\ \gamma_{ij} \cap (\sim(\alpha_i\ \beta_i^{\ *})) =_s \phi)\ \underline{\text{and}}\ (\delta_j (\sim \gamma_{ij}) \cap \alpha_i \beta_i^{\ *} =_s \phi))$$

$$S_4 \equiv (\forall i \in (n))\ (\forall j \in (m))\ ((\beta_i^{\ +}, \gamma_{ij}) \sim \xi_{ij})$$

$$S_5 \equiv (\forall j \in (m))\ (\eta_j =_s \beta_0^{\ +} \cap (\sim \sum_{i=1}^{n} \xi_{ij}))$$

$$S_6 \equiv (\forall j \in (m))\ (\text{cf } (\eta_j))$$

The family B_1 of the inclusion rules is the enumerable set of inference rules which is generated by the inclusion scheme B1 :

$$\frac{S_0\ ,\ S_1\ ,\ S_2\ ,\ S_3\ ,\ S_4\ ,\ S_5\ ,\ S_6}{\alpha_0\ \beta_0^{\omega} \subseteq \sum_{i=1}^{n} \alpha_i\ \beta_i^{\omega}} \qquad \text{(B1)}$$

The conjugation rules (B_2)

Let S stand for the following scheme of (sets of) formulas :

$S \equiv (\forall i \in [n])$ (ne (ξ_i) and $\xi_i\ \eta_i \cap (\sim \beta) =_s \phi$ and $\xi_i\ (\sim\eta_i) \cap \beta =_s \phi$)

The family B_2 of the conjugation rules is the enumerable set of inference rules which is generated by the inclusion scheme B2 :

$$\frac{\text{ne }(\beta),\ \alpha\ \pi\ \beta\ ,\ \alpha + \beta' =_s \sum_{i=1}^{n} \xi_i\ ,\ \delta =_s \sum_{i=1}^{n} (\eta_i \cap \gamma)\ \xi_i\ ,\ S}{(\beta\ ,\ \gamma) \curvearrowright \delta} \qquad \text{(B2)}$$

The disjunction rules (B_3)

Let T stand for the following scheme of (sets of) formulas : $T \equiv$ $(\forall i \in [1,2])$ (ne (α_i) and $(\alpha_i \cap \phi^* =_s \phi$ or $\alpha_i =_s \phi^*)$ and ne (β_i) and $(\beta_i \cap \phi^* =_s \phi))$. The family B_3 of the disjunction rules is the enumerable set of inference rules which is generated by the disjunction scheme B3 :

$$\frac{(\alpha_2\ \beta_2^+,\ \alpha_1\ \beta_1^+)\ \%\ \delta\ ,\ (\beta_2^+,\ \delta) \curvearrowright \gamma\ ,\ \beta_1^+ \cap \gamma =_s \phi\ ,\ T}{\alpha_1\ \beta_1^{\omega} \oplus \alpha_2\ \beta_2^{\omega}} \qquad \text{(B3)}$$

The division rules (B_4)

Let T_1 to T_3 be the following schemes of sets of formulas :

$T_1 \equiv (\forall i \in (n))$ (ne (ξ_i) and $\xi_i\, \eta_i \cap (\sim \beta) =_s \phi$ and $\xi_i\, (\sim \eta_i) \cap \beta =_s \phi)$

$T_2 \equiv (\forall i \in (n))$ $(\xi_i \cap \gamma =_s \mu_i)$

$T_3 \equiv (\forall i \in (n))$ $((\mu_i =_s \phi$ and $\delta_i =_s \phi)$ or (ne(μ_i) and $\delta_i =_s \eta_i))$

The family B_4 of the division rules is the enumerable set of inference rules which is generated by the division scheme B4 :

$$\frac{\text{ne}\ (\beta),\quad \alpha\ \pi\ \beta,\ \alpha + \beta' =_s \sum_{i=1}^{n} \xi_i\ ,\ \delta =_s \sum_{i=1}^{n} \delta_i\ ,\ T_1\ ,T_2\ ,\ T_3}{(\beta\ ,\gamma)\ \%\ \ \delta} \qquad \text{(B4)}$$

The prefix rules (B_5)

B_5 is the finite set of the following inference rules B5.1 to B5.5 :

$$\frac{\alpha =_s \phi}{\phi\ \ \pi\ \ \alpha} \qquad \text{(B5.1)}$$

$$\frac{\alpha =_s x_j}{\phi^*\ \ \pi\ \ \alpha} \qquad \text{for any } j \text{ in } (r)\ \text{(B5.2)}$$

$$\frac{\alpha\ \pi\ \alpha',\ \beta\ \pi\ \beta'}{\alpha + \beta\ \ \ \pi\ \alpha' + \beta'} \qquad \text{(B5.3)}$$

$$\frac{\alpha\ \pi\ \alpha'\ ,\ \beta\ \pi\ \beta'\ ,\ \text{ne}\ (\beta')}{\alpha + \alpha'\beta\ \ \ \pi\ \alpha'\ \beta'} \qquad \text{(B5.4)}$$

$$\frac{\alpha\ \pi\ \alpha'}{\alpha'^*\alpha\ \ \pi\ \ \alpha'^*} \qquad \text{(B5.5)}$$

The cycle-free rules (B_6)

B_6 is the finite set of the following axioms and inference rules B6.1 to B6.4 :

$$\mathrm{cf}\ (\phi) \qquad \text{(B6.1)}$$

$$\frac{\mathrm{cf}\ (\alpha)\ ,\ \mathrm{cf}\ (\beta)}{\mathrm{cf}\ (\alpha+\beta)} \qquad \text{(B6.2)}$$

$$\frac{\mathrm{pf}\ (\alpha),\ \mathrm{pf}\ (\beta),\ \ \alpha\beta^{*} \cap \beta^{*} =_{s} \phi}{\mathrm{cf}\ (\alpha\beta^{*})} \qquad \text{(B6.3)}$$

$$\frac{\alpha\ \pi\ \beta\ ,\ \alpha \cap \beta =_{s} \phi}{\mathrm{pf}\ (\beta)} \qquad \text{(B6.4)}$$

We have now finished the presentation of the rules, and are ready to state the final result of the paper which is claimed by the following

Theorem $A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$

is a consistant and complete (relative to $=_s$) set of axioms and inference rules for ($\mathbb{W}_{I_r} + I''$, {ne, new, pf, cf , = , $\subseteq$, $\oplus$, $\sim$, $\%$ })
(the proof is straightforward and is left to the reader).

7. FINAL COMMENTS.

Our initial purpose was to construct a formal proof system for infinitary rational languages, taking equalities between finitary rational languages as an infinite set of axioms. Last theorem shows that we have to some extent succeeded in that task. However, our system deserves at least two kinds of critical remarks : first, auxiliary properties and relations have been introduced beside the relations of interest (equality and disjointness), and secondly, a finite proof system has not been reached (even after excluding the axioms of the equality between finitary languages) since schemes of inference rules have been appealed to. Our system has altogether the merit to exist. It is not clear for the moment whether the inconveniences acknowledged above are inherent to infinitary languages, at least when complementation of languages has to be accounted for, or whether they are drawbacks of the approach which consists in drawing the proof system from a corresponding (yet implicit) decision procedure. Much remains to be done for answering the above questions. Other issues are the examination of the decision procedure w.r.t. complexity, and the investigation of the class of periodic languages of the form $\sum_i \alpha_i \beta_i^* (\beta_i^+)^\Omega$, for which the equality is a decidable property.

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APPENDIX

proof of proposition 9.

. only if part . Let j laying between 1 and m, one has to prove $(C^+)^\Omega \subseteq \sum_i X_{ij} C_i^* (C_i^+)^\Omega$ from the assumption $BC^*(C^+)^\Omega \subseteq \sum_i B_i C_i^* (C_i^+)^\Omega$.

According to the definition of the operator ÷ , one can find some appropriate word ϕ in BC^* such that $X_{ij} = B_i C_i^* / \phi$ for any i laying between 1 and n. Now, $\phi(C^+)^\Omega$ is included in $\sum_i B_i C_i^* (C_i^+)^\Omega$, so that the result comes by direct application of the general equality $EF^*(F^+)^\Omega / \phi = (EF^* / \phi) F^*(F^+)^\Omega$ under the assumptions $E,F \in Rat(I_r^*)$, $\phi \in I_r^*$, $\lambda \notin F$. Let us demonstrate the above equality.

i) Given ψ in $EF^*(F^+)^\Omega / \phi$, there exist $u \in E$, $v \in F^*$ and $w \in F^+$ such that $\phi\psi = u v w^\omega$. Since ϕ is a finite word and since $\lambda \notin F$, there must exist some n such that $\overline{\phi} \leqslant \overline{u v w^n}$. Put $u v w^n = \phi\psi'$, then obviously $\psi' \in (EF^*/\phi) F^*$, and thus $\psi \in (EF^*/\phi) F^*(F^+)^\Omega$, for ψ is equal to $\psi' w^\omega$.

ii) Given ψ in $(EF^* / \phi) F^*(F^+)^\Omega$, ψ obviously belongs to $EF^* (F^+)^\Omega / \phi$.

. if part. Let ϕ in BC^*, it is sufficient to prove the inclusion $\phi(C^+)^\Omega \subseteq \sum_i B_i C_i^* (C_i^+)^\Omega$ from the assumption $(C^+)^\Omega \subseteq \sum_i X_{ij} C_i^*(C_i^+)^\Omega$ for any j between 1 and m. According to the definition of the operator ÷ , there must exist some j, $1 \leqslant j \leqslant m$, such that $\langle B_1C_1^*, \ldots, B_nC_n^* \rangle / \phi = \langle X_{1j}, \ldots, X_{nj} \rangle$. Since the inclusion $\phi X_{ij} \subseteq B_i C_i^*$ holds for any i, the desired result directly comes from the assumption, remarking that $C_i^* C_i^* = C_i^*$ ■

proof of proposition 10.

It is easily seen that the condition is necessary, thus it is enough to suppose that there exists some word w in the intersection $BC^*(C^+)^\Omega \cap B'C'^*(C'^+)^\Omega$ and to show that $(C^+)^\Omega$ and $(B'C'^* \,/\!.\, BC^*)(C'^+)^\Omega$ cannot be disjoint in that case. From the hypothesis, there exist $\phi \in BC^*$, $\phi' \in B'C'^*$, $\Theta \in C'$, $\Theta' \in C'^+$ such that $\phi\Theta^\omega = w = \phi'\Theta'^\omega$. Since ϕ is a finite word and since $\lambda \notin C^+$, there must exist some n such that $\overline{\phi} \leqslant \overline{\phi'\Theta'^n}$. Put $\phi\psi = \phi'\Theta'^n$, then $\psi \in (B'C'^*) \,/\!.\, (BC^*)$ and $\Theta^\omega = \psi\Theta'^\omega$: clearly, $(C^+)^\Omega$ and $(B'C'^* \,/\!.\, BC^*)(C'^+)^\Omega$ have a non-empty intersection.

■

proof of proposition 11.

. only if part. Let ψ in C^+, it is sufficient to prove $\psi^\omega \varepsilon (C_i^+ \curvearrowright A_iC_i^*)^\Omega$ from the assumption $\psi^\omega \varepsilon A_iC_i^*(C_i^+)^\Omega$. By the assumption, there exist ϕ' in $A_iC_i^*$ and ψ' in C_i^+ such that $\psi^\omega = \phi'\psi'^\omega$. Notice that neither ψ nor ψ' can be equal to λ. Define Θ as the least non-empty word which may be written as $\Theta = \phi'\ \psi'^n$ for some $n \geqslant 0$. Define Θ' as the least non empty word such that $\Theta\Theta' = \psi^m$ holds for some $m > 0$. The following equalities are verified :

$\Theta\ \psi'^\omega = \psi^\omega = \Theta\Theta'\ \psi^\omega$, hence $\psi'^\omega = \Theta'\Theta\ \psi'^\omega = (\Theta'\Theta)^\omega$.

As a consequence, there must exist strictly positive integers k and l such that $(\Theta'\Theta)^k = \psi'^l$.

Put $\Theta'' = \Theta\ \psi'^l = \phi'\ \psi'^{l+n}$. The following equalities hold :

$$\Theta''\Theta' = \Theta\ (\Theta'\Theta)^k\ \Theta' = (\Theta\ \Theta')^{k+1} = \psi^{m\ (k+1)}$$

$$\Theta'\Theta'' = \Theta'\Theta\ \psi'^l = (\Theta'\Theta)^{k+1}$$

Obviously, $(\Theta'\Theta'')^\omega = (\Theta'\Theta)^\omega = \psi'^\omega$, and there must exist strictly positive integers p and q such that $(\Theta'\Theta'')^p = \psi'^q$. Put $\Theta''' = (\Theta'\Theta'')^{p-1}\ \Theta'$, then $\Theta'''\Theta'' = (\Theta'\Theta'')^p = \psi'^q$, and $\Theta''\Theta''' = (\Theta''\Theta')^p = \psi^{m(k+1)p}$.

$\psi' \varepsilon C_i^+,\ \phi' \varepsilon A_iC_i^* \Rightarrow \Theta'' \varepsilon A_iC_i^*$ (since $\Theta'' = \phi'\ \psi'^{l+n}$) ;

$\psi' \varepsilon C_i^+ \Rightarrow \Theta'''\Theta'' \varepsilon C_i^+$ (since $\Theta'''\Theta'' = \psi'^q$) ;

hence $\Theta''\Theta''' \varepsilon (C_i^+ \curvearrowright A_iC_i^*)$.

The desired result now follows by simply remarking $(\Theta''\Theta''')^\omega = \psi^\omega$.

. if part. It is sufficient to prove the inclusion

$$(C_i^+ \curvearrowright A_iC_i^*)^\Omega \subseteq A_iC_i^*\ (C_i^+)^\Omega.$$

Let ψ be a word in $C_i^+ \curvearrowright A_iC_i^*$. According to the definition of the operator $\curvearrowright$, there exist Θ in $A_iC_i^*$ and θ' in I_r^* such that $\psi = \Theta\ \Theta'$ and $\Theta'\Theta\ \varepsilon\ C_i^+$. Now $\psi^\omega = \Theta\ (\Theta'\Theta)^\omega$, and thus $\psi^\omega\ \varepsilon\ A_iC_i^*\ (C_i^+)^\Omega$. ■

proof of proposition 12.

The only-if part of the proof is a direct application of the inclusion $(C'^+ \curvearrowright A'C'^*)^\Omega \subseteq A'C'^*\ (C'^+)^\Omega$, which has been established just above. In order to demonstrate the reverse implication, suppose that there exists some word ψ in C^+ such that $\psi^\omega\ \varepsilon\ A'C'^*(C'^+)^\Omega$, and let us show that ψ^ω then belongs to $(C'^+ \curvearrowright A'C'^*)^\Omega$.

By our assumptions, $\{\psi^\omega\} = (\psi^+)^\Omega \subseteq A'C'^* (C'^+)^\Omega$. Applying proposition 11. one obtains $\{\psi^\omega\} \subseteq (\psi^+ \cap (C'^+ \sim A'C'^*))^\Omega$, that is still $\psi^\omega \in (C'^+ \sim A'C'^*)^\Omega$. ■

proof of proposition 14.

The condition is easily proved sufficient, since it implies that for any ψ in C^+, there exists some strictly positive p such that $\psi^p \in (\sum_i C_i^+ \sim A_iC_i^*)$. In order to establish the reverse implication, let us suppose that the condition does not hold, i.e. $(\exists\ \phi \in C^+)\ (\phi^+ \cap (\sum_i C_i^+ \sim A_iC_i^*) = L_\phi)$. If the inclusion $(C^+)^\Omega \subseteq (\sum_i (C_i^+ \sim A_iC_i^*))^\Omega$ nevertheless holds, then ϕ^ω equals ψ^ω for some ψ in $\sum_i (C_i^+ \sim A_iC_i^*)$, and thus one can find strictly positive integers p and q such that $\phi^p = \psi^q$. Now, it is easily seen that the language $\sum_i (C_i^+ \sim A_iC_i^*)$ contains all and every strictly positive powers of any of its words. Hence $\psi^q \in \sum_i (C_i^+ \sim A_iC_i^*)$, which contradicts the assumption $\phi^+ \cap \sum_i (C_i^+ \sim A_iC_i^*) = L_\phi$. ■

proof of proposition 15.

The condition is obviously necessary. In order to prove that it is also sufficient, let us suppose that there exists some word ψ in C^+ s.t. $\psi^\omega \in (C'^+ \sim A'C'^*)^\Omega$. Clearly, one can find ϕ in $C'^+ \sim A'C'^*$ and strictly positive integers p and q which verify $\phi^p = \psi^q$. Now $C'^+ \sim A'C'^*$ does contain every strictly positive powers of ϕ, hence $\phi^p \in C'^+ \sim A'C'^*$ and $C^+ \cap (C'^+ \sim A'C'^*)$ is not empty. ■

proof of proposition 21.

Since X is a co-residue of L, there exists some residue X' of L such that X and X' are L-correlated. Let $\mathcal{A}$ be a finite deterministic automaton such that $|\mathcal{A}| = L$. Put $\mathcal{A} = (Q,f,q_0,Q_f)$, and define Q'_f as the subset of states q' in Q such that $|(Q,f,q',Q_f)| = X'$. Obviously, X is equal to $|(Q,f,q_0,Q'_f)|$, and thus X is a rational language. ■

proof of proposition 24.

. only if part. For each particular i, let the residual normal form of B_i be given as $\sum_{k=1}^{n_i-1} Y_{ik}\, Y'_{ik}$. Let $Y_{in_i} = \sim (\sum_{k=1}^{n_i-1} Y_{ik})$ and $Y'_{in_i} = L_\phi$, then $\{Y_{i1}, \ldots, Y_{in_i}\}$ is a partition of I_r^* and B_i is still equal to the extended sum

$\sum_{k=1}^{n_i} Y_{ik} Y'_{ik}$. According to the definition of $\div$, $<B_1, \ldots, B_n> \div B$ is the set of vectors $<Y'_{1j_1}, \ldots, Y'_{nj_n}>$ such that the index values j_i verify (4) and (5) :

$$(\forall i)\ (1 \leqslant j_i \leqslant n_i) \qquad (4)$$

$$B \cap (\bigcap_i Y_{ij_i}) \neq L_\phi \qquad (5)$$

Define J as the set of n-tuples $<j_1, \ldots, j_n>$ which verify (4) and (5). Let m be the cardinality of J and let f be some bijection between J and $\{1, \ldots, m\}$. For $\overline{j} = <j_1, \ldots, j_n>$ in J, define $Z'_{\overline{j}}$ equal to $B \cap (\bigcap_i Y_{ij_i})$ and $Z_{f(\overline{j})}$ equal to $Z'_{\overline{j}}$.

Since the co-residues of a rational language are rational, this property is also true of Z_j' s, which obviously satisfy conditions 1 and 2 of the proposition. Now put $X_{i\,f(\overline{j})} = Y'_{ij_i}$ for $\overline{j}$ as above, then the third condition is also satisfied, since either $Z_{f(\overline{j})}$ is included in Y_{ij_i} which is the co-residue correlated with $X_{if(\overline{j})}$ in B_i , or $Z_{f(\overline{j})}$ is included in Y_{in_i} which contains no left factor of words in B_i and $X_{if(\overline{j})}$ is empty.

. <u>if part</u>. It is easily seen from conditions 1 and 3 that for any i, the X_{ij} 's are residues of the corresponding B_i , and that for any j, B_i/ϕ equals X_{ij} for any ϕ in Z_j. The proof follows by condition 2. ■

proof of proposition 27.

The condition is obviously necessary. Let us show that it is also sufficient. Since cf(X) and cf(Y) imply $\lambda \notin X + Y$, one is left with proving the following property : $(\forall\ \phi \in X + Y)\ ((X + Y) \cap \phi^+ \neq \phi^+)$. Owing to the implications $cf(X) \supset cf(X \cap \phi^+)$ and $cf(Y) \supset cf(Y \cap \phi^+)$, it suffices to prove $cf(X \cap \phi^+) \wedge cf(Y \cap \phi^+) \wedge \phi \neq \lambda \supset ((X + Y) \cap \phi^+) \neq \phi^+$, or still equivalently :

$$cf(X \cap \phi^+) \wedge cf(Y \cap \phi^+) \wedge \phi \neq \lambda \supset ((X \cap \phi^+) + (Y \cap \phi^+)) \neq \phi^+.$$

Define I' as the singleton set $\{y\}$, where letter y does not belong to I_r. Given ϕ in I_r^+ , let $\eta_\phi : I'^* \to I_r^*$ be the monoid homomorphism which verifies $\eta_\phi(y) = \phi$, then for any L in $Rat(I_r^*)$, $L \cap \phi^+$ is the morphic image $\eta_\phi(L')$ of a corresponding language L' in $Rat(I'^*)$. Since there exists a unique L' which verifies $\eta_\phi\ (L') = L \cap \phi^+$ for given L and ϕ , we can freely denote L' by

$\eta_\phi^{-1}(L \cap \phi^+)$. It is easily seen that for any L in Rat(I_r^*) and ϕ in I_r^+, cf $(L \cap \phi^+)$ holds iff cf $(\eta_\phi^{-1})(L))$ also holds. Let $X' = \eta_\phi^{-1}(X \cap \phi^+)$ and $Y' = \eta_\phi^{-1}(Y \cap \phi^+)$. Using the above remark, it is enough to prove the following implication for $\phi \neq \lambda$: $cf(X') \wedge cf(Y') \supset (X' + Y') \neq y^+$. If at least one of languages X', Y' is a finite set of words, that property is immediate. In any other cases and due to the construction of languages X' and Y', one can always find expressions *X* and *Y* of respective forms

$$X \equiv y^{n_1} + y^{n_2} + \ldots + y^{n_i} (y^{m_j})^* (\phi^* + y^{m_1} + y^{m_2} + \ldots + y^{m_{j-1}})$$

$$Y \equiv y^{p_1} + y^{p_2} + \ldots + y^{p_k} (y^{q_1})^* (\phi^* + y^{q_1} + y^{q_2} + \ldots + y^{q_{1-1}})$$

which verify $X' = |X|$ and $Y' = |Y|$,

with $0 < n_1 < n_2 < \ldots < n_i$, $0 < m_1 < m_2 < \ldots < m_j$

and $0 < p_1 < p_2 < \ldots < p_k$, $0 < q_1 < q_2 < \ldots < q_1$.

Let us assume for a moment that $(X' + Y')$ equals y^+, and show that this supposition is nonsense.

Let the integers g and h such that $g < m_j$ and $n_i = hm_j + g$, then $n_i + (m_j - g) = (h + 1)m_j$.
If $g = 0$, then $(y^{n_i})^+$ is included in X' and the hypothesis cf(X') is therefore contradicted.
If $m_j - g$ belongs to the set $\{m_1, m_2, \ldots, m_{j-1}\}$, the hypothesis cf (X') is contradicted again by the inclusion $(y^{n_i+m_j-g})^+ \subseteq X'$. Now exclude the above situations, then no strictly positive power of the word $y^{n_i+m_j-g}$ belongs to X' since, for $s > 0$, $(y^{n_i+m_j-g})^s$ is equal to $y^{n_i}\ y^{(s-1)(h+1)m_j}\ y^{m_j-g}$. From the assumption $X' + Y' = y^+$, it follows that every strictly positive powers of the word $y^{n_i+m_j-g}$ belong to Y', so that the hypothesis cf(Y') is contradicted.
Thus, $cf(X') \wedge cf(Y') \supset (X' + Y') \neq y^+$ as it was to prove, and one can conclude that cf(X) and cf(Y) imply cf (X + Y). ■

proof of proposition 28.

The condition is obviously necessary. Let us prove that it is also sufficient. Suppose for a moment that there exists some word ϕ in I_r^* such that $\phi^+ \subseteq XY^*$, and show that this supposition is nonsense. Put $\phi = \psi\psi'$ with $\psi \varepsilon X$ and $\psi' \varepsilon Y^*$. Since X is prefix-free, ψ is necessarily different from λ, and one can therefore assume $\psi' \neq \lambda$ without loss of generality : for $\psi' = \lambda$,$\psi \varepsilon X$ and $\psi^2 \varepsilon XY^*$ imply $\psi \varepsilon Y^*$ by prefixity of X, so that there remains to consider $\phi' = \psi^2$. Let us assume $\psi' \neq \lambda$. $\phi^+ \subseteq XY^*$ implies $\psi\psi'\psi\psi' \varepsilon XY^*$, and thus $\psi'\psi\psi' \varepsilon Y^*$ by the prefixity of X. Put $\psi' = \psi_1\psi_2\ldots\psi_n$, with $\psi_i \varepsilon Y$ for any i. Using the prefixity of Y, it may be shown by induction on i that $(\psi_i \ldots \psi_n\psi\psi')/\psi_i$ belongs to Y^* for any i in$\{1\ldots n\}$. In particular, $\psi\psi'$ belongs to Y^* , which contradicts the hypothesis $XY^* \cap Y^* = L_\phi$. ■

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